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BAYES ESTIMATION IN THE INVERSE RAYLEIGH MODEL

Gyan Prakash*

Department of Community Medicine, S. N. Medical College, Agra, U. P., India

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***Abstract**: The properties of Bayes estimates of the parameter, reciprocal of the parameter, reliability function and hazard rate have been studied for the inverse Rayleigh model under two different loss functions in the present paper. We also predict the future order statistic based on the observed ordered statistic and obtain the prediction intervals for unobserved order statistic under One and Two–Sample prediction technique.*

***Keywords**: Bayes estimate, reliability function, hazard rate and prediction interval.*

1. Introduction

If x be the random variable said to have follow the inverse Rayleigh distribution with the parameter θ, has the distribution function

$$F(x;\theta) = \exp\left(-\frac{\theta}{x^2}\right); x > 0, \theta > 0. \tag{1}$$

Let $x_1, x_2, \ldots, x_n$ be the n random observations drawn from the model (1) then maximum likelihood estimator (MLE) of the parameter θ is $\hat{\theta}_{ML} = \frac{n}{T}; T = \sum_{i=1}^{n} \frac{1}{x_i^2}$. Also, the reliability function $\psi(t)$ and the hazard rate $\rho(t)$ for a specific mission time $t\,(>0)$ are obtained as:

$$\psi(t) = 1 - e^{-\frac{\theta}{t^2}} \tag{2}$$

* Email: ggyanji@yahoo.com

and

$$\rho(t) = \frac{2\theta}{t^3}\left(\exp\left(\frac{\theta}{t^2}\right) - 1\right)^{-1}. \tag{3}$$

In the Bayesian estimation problem when positive and negative errors have different consequences, the use of SELF (squared error loss function) is not appropriate. Varian [19] had discussed an asymmetric loss function known as the LINEX loss function (LLF). This loss function is convex and its shape is determined by the value of its shape parameter. The positive (negative) values of the shape parameter, gives more weight to underestimation (overestimation). In addition, the magnitude of the shape parameter reflects the degree of asymmetry. The LLF is defined (when $\hat{\theta}$ be any estimate of the parameter θ) as:

$$L(\Delta) = e^{a\Delta} - a\Delta - 1\ ; a \neq 0 \text{ and } \Delta = (\hat{\theta} - \theta). \tag{4}$$

Here 'a' is the shape parameter of the LLF. When $a > 0$, the loss function increases almost exponentially for positive Δ and almost linearly otherwise and overestimation is more heavily penalized than underestimation. When $a < 0$, the linear exponential rises are interchanged and underestimation is considered more costly than overestimation. The LLF may be considered a natural extension of SELF (for small values of 'a' (near to zero) the LLF tends to SELF). Srivastava & Tanna [16], Xu & Shi [20], Prakash & Singh [9], Singh et al. [11], Prakash & Singh [10] and others have discussed recently the estimation procedures under LLF.

Soland [14] has been studied Bayesian analysis for the Weibull process with unknown scale and shape parameters. Banerjee & Bhattacharya [3] have studied the application of the inverse Gaussian distribution under Bayesian results. Zellner [21], Sinha [13], Fernandez [5], Raqab & Madi [18], Mousa & Al–Sagheer [7], Son & Oh [15], Ahmad et al. [1] are few of those who have studied the properties of the estimators under Bayesian setup. An important objective of a life–testing experiment is to predict the nature of the future sample based on current sample. Howlader [6] derived the highest posterior density (HPD) prediction intervals for the kth order statistic in a future sample. Raqab [17] discussed the prediction problems for the Rayleigh and normal models. Bain [2], Sinha [12], Cramer & Kamps [4], Nigm et al. [8] and Ahmed et al. [1] are few of those who have been extensively studied predictive inference for future observations.

The conjugate prior density of the parameter θ is considered as the two parameter Gamma distribution with parameter (α, β) and the posterior density are given as:

$$g(\theta) = \frac{\beta^{\alpha}}{\Gamma\alpha}\theta^{\alpha-1} e^{-\beta\theta}\ ; \theta > 0, \alpha > 0, \beta > 0. \tag{5}$$

and

$$Z(\theta) = \frac{\left(T+\beta\right)^{n+\alpha}}{\Gamma(n+\alpha)}\theta^{n+\alpha-1}\ e^{-(T+\beta)\theta}\ ; \theta > 0. \tag{6}$$

This paper suggests some Bayes estimators for the parameter, reciprocal of the parameter, reliability function and hazard rate under the natural conjugate prior density with respect to the symmetric and asymmetric loss functions. The properties in terms of risk and Bayes risk have been studied by the simulation study. The prediction intervals of the future observations are also determined under the One –Sample and Two–Sample prediction techniques.

2. The Bayes Estimators of the parameter θ

The Bayes estimate of θ under the SELF is obtained as:

$$\hat{\theta}_1 = E_P(\theta) = \frac{n+\alpha}{T+\beta}.$$

Here, P indicates the expectation is taken under the posterior density. Similarly, the Bayes estimate of θ under the LLF (1.4) is obtained as:

$$\hat{\theta}_2 = -\frac{1}{a}\ln\left(E_P\left(e^{-a\theta}\right)\right) = \frac{n+\alpha}{a}\ln\left(1+\frac{a}{T+\beta}\right).$$

The expressions of the risk and the Bayes risk are summarized in the following table 1 under both risks criterions:

Table 1. Estimator and Their Risk

Estimator	Risk
$\hat{\theta}_i$	$R_{(S)}\left(\hat{\theta}_i\right) = G\left(\Delta_i^2 - 2\theta\Delta_i\right) + \theta^2;\ \Delta_1 = (n+\alpha)\left(\frac{z}{\theta}+\beta\right)^{-1}$
	$R_{(BS)}\left(\hat{\theta}_i\right) = I\left(\Delta_i^2 - 2\theta\Delta_i\right) + \frac{b(b+1)}{\beta^2}\ ;i = 1, 2$
	$R_{(L)}\left(\hat{\theta}_i\right) = G\left(e^{\Delta'_i} - \Delta'_i\right) - 1;\ \Delta'_i = a\left(\Delta_i - \theta\right)$
	$R_{(BL)}\left(\hat{\theta}_i\right) = I\left(e^{\Delta'_i} - \Delta'_i\right) - 1;\ \Delta_2 = \left(\frac{n+\alpha}{a}\right)\ln\left(1+a\left(\frac{z}{\theta}+\beta\right)^{-1}\right)$

where $I(w) = \beta^{\alpha}\int_{\theta=0}^{\infty}\left(\int_{z=0}^{\infty}\frac{e^{-z}z^{n-1}(w)\,dz}{\Gamma n\,\Gamma\alpha}\right)e^{-\beta\theta}\theta^{\alpha-1}d\theta$, $G(w) = \int_0^{\infty}\frac{e^{-z}z^{n-1}(w)}{\Gamma n}dz$ and w be the function of z and θ both. Here, the suffix S and L indicates respectively the risks taken under the SELF and the LLF criterion. Similarly, the suffix BS and BL denotes the Bayes risks under corresponding risk criterion.

3. The Bayes Estimators of the parameter θ^{-1}

The Bayes estimate of θ^{-1} under the SELF is:

$$\hat{\theta}_3 = \omega_1 \left(T + \beta \right) ; \omega_1 = \left(n + \alpha - 1\right)^{-1} .$$

The Bayes estimate of θ^{-1} under the LLF (4) does not exist. Hence, we consider the invariant version of the LINEX loss function (ILLF) and is defined for θ as:

$$L(\Delta') = e^{a\Delta'} - a\Delta' - 1 \; ; \; \Delta = \frac{\hat{\theta} - \theta}{\theta} .$$ (Singh et al. [11])

and the Bayes estimate of θ^{-1}, $\hat{\theta}_4$ (say) is obtained by solving the given equality:

$$E_P\left(\theta e^{a\hat{\theta}_4\theta}\right) = e^a E_P(\theta) \Rightarrow \hat{\theta}_4 = \omega_2 \left(T + \beta\right) ; \omega_2 = \frac{1}{a}\left(1 - e^{-a/(n+\alpha-1)}\right).$$

The expressions of the risks and the Bayes risk under the SELF and the ILLF are summarized in the following table 2:

Table 2. Bayes Risk under the SELF and the ILLF

Estimator	Risk
$\hat{\theta}_j$	$R_{(S)}\left(\hat{\theta}_j\right) = \frac{n\omega_i^2}{\theta^2} + \left\{\omega_i\beta + \frac{n\omega_i - 1}{\theta}\right\}^2$
	$R_{(BS)}\left(\hat{\theta}_j\right) = \beta^2\left\{\frac{\left(n\omega_i - 1\right)^2 + n\omega_i^2}{(\alpha-1)(\alpha-2)} - 2\omega_i\frac{\left(n\omega_i - 1\right)}{(\alpha-1)} + \omega_i^2\right\}$
	$R_{(L)}\left(\hat{\theta}_J\right) = \frac{\exp\left(a\omega_i\beta\theta - a\right)}{\left(1 - a\omega_i\right)^n} - a\omega_i\left(n - \beta\theta\right) + a - 1$
	$R_{(BL)}\left(\hat{\theta}_j\right) = \frac{\exp\left(-a\right)}{\left(1 - a\omega_i\right)^{n+\alpha}} - a\omega_i\left(n + \alpha\right) + a - 1, i = 1, 2, j = 3, 4$

4. Numerical Analysis

The expressions for the risk and Bayes risk of the estimators $\hat{\theta}_i\,; i = 1,\ldots,4$ involve n, a, θ, α and β. In order to study the performances of the estimators, a simulation study has been carried out. For this, we draw 10,000 samples of sizes $n = 05, 10, 15$ with the given parametric values of $a = 0.50, 1.00, 1.50$; $\theta = 04, 06, 08$ and $(\alpha, \beta) = (4.00, 2.00), (9.00, 3.00)$ for the given model (1). The values of the prior parameters α and β meets the criterion that the prior variance should be unity. The numerical findings are presented in the Tables 3–4 only for $n = 05, 10$ and $\theta = 04$.

Table 3 shows that the risks and Bayes risks decrease when the sample size n increases when other parametric values are fixed for the estimators $\hat{\theta}_1$ and $\hat{\theta}_2$. The risks also increase (decrease) when $\theta\ (\alpha, \beta)$ increases for these estimators. The Bayes risks of these estimators increase when prior parameter (α, β) increases.

The risk and Bayes risks both also increase when $'a'$ increases for $\hat{\theta}_2$ under SELF and the LLF–criterion both but for the estimator $\hat{\theta}_1$ only under LLF–criterion with other fixed parametric values. In addition, the magnitude of the risk and the Bayes risk is larger under the SELF–criterion with respect to the LLF.

Both the risks and Bayes risks decrease when the sample size n or prior parameter (α, β) increases for other fixed parametric values of the estimators $\hat{\theta}_3$ and $\hat{\theta}_4$(Table 4). The risks increase when θ increases for both estimators.

The risk and Bayes risks increase when $'a'$ increases for $\hat{\theta}_4$ under SELF and LLF–criterion (except risk under SELF-criterion) whereas the estimator $\hat{\theta}_3$ only under LLF–criterion with other fixed parametric values. The magnitude of the Bayes risk is larger under SELF–criterion with respect to LLF–criterion only for $\alpha = 2.25$ and $\beta = 1.50$.

5. The Bayes Estimator of Reliability Function and Hazard Rate

The Bayes estimate of the reliability function under the SELF, corresponding to the posterior density $Z(\theta)$ is given as:

$$\psi_1 = 1 - \left(1 + t^{-2} + (T + \beta)^{-1}\right)^{-n-\alpha}.$$

The Bayes estimate of the hazard rate under the SELF does not exist in closed form. However, one may obtain it numerically by solving the given expression:

Table 3. Risk for the Bayes estimators $\hat{\theta}_1$ and $\hat{\theta}_2$ under the SELF and the LLF.

$\theta = 04$		n = 05			n = 10		
a ↓	$\beta \rightarrow$	1.50	2.00	3.00	1.50	2.00	3.00
	$\alpha \rightarrow$	2.25	4.00	9.00	2.25	4.00	9.00
0.50	$R_{(S)}(\hat{\theta}_1)$	1.8636	1.5416	0.5963	1.0564	0.9352	0.4682
	$R_{(BS)}(\hat{\theta}_1)$	0.3943	0.4990	0.6555	0.2453	0.3335	0.4896
	$R_{(L)}(\hat{\theta}_1)$	0.1816	0.1535	0.0632	0.1080	0.0963	0.0507
	$R_{(BL)}(\hat{\theta}_1)$	0.0467	0.0596	0.0800	0.0297	0.0406	0.0598
1.00	$R_{(S)}(\hat{\theta}_1)$	1.8636	1.5416	0.5963	1.0564	0.9352	0.4682
	$R_{(BS)}(\hat{\theta}_1)$	0.3943	0.4990	0.6555	0.2453	0.3335	0.4896
	$R_{(L)}(\hat{\theta}_1)$	0.5870	0.5042	0.2184	0.3649	0.3258	0.1792
	$R_{(BL)}(\hat{\theta}_1)$	0.1929	0.2463	0.3434	0.1230	0.1682	0.2509
1.50	$R_{(S)}(\hat{\theta}_1)$	1.8636	1.5416	0.5963	1.0564	0.9352	0.4682
	$R_{(BS)}(\hat{\theta}_1)$	0.4838	0.6161	0.9053	0.3059	0.4176	0.6366
	$R_{(L)}(\hat{\theta}_1)$	1.0986	0.9549	0.4304	0.7140	0.6338	0.3630
	$R_{(BL)}(\hat{\theta}_1)$	0.3943	0.4990	0.6555	0.2453	0.3335	0.4896
0.50	$R_{(S)}(\hat{\theta}_2)$	2.4361	2.0118	0.8426	1.3302	1.1833	0.6022
	$R_{(BS)}(\hat{\theta}_2)$	0.4093	0.5173	0.6856	0.2526	0.3431	0.5077
	$R_{(L)}(\hat{\theta}_2)$	0.2334	0.1971	0.0883	0.1334	0.1201	0.0640
	$R_{(BL)}(\hat{\theta}_2)$	0.0448	0.0571	0.0771	0.0288	0.0393	0.0580
1.00	$R_{(S)}(\hat{\theta}_2)$	2.9874	2.4740	1.1165	1.6362	1.4557	0.7672
	$R_{(BS)}(\hat{\theta}_2)$	0.4434	0.5611	0.7554	0.2706	0.3677	0.5521
	$R_{(L)}(\hat{\theta}_2)$	0.8854	0.7629	0.3899	0.5301	0.4824	0.2760
	$R_{(BL)}(\hat{\theta}_2)$	0.1650	0.2109	0.2948	0.1087	0.1486	0.2222
1.50	$R_{(S)}(\hat{\theta}_2)$	3.5022	2.9153	1.4024	1.9533	1.7378	0.9515
	$R_{(BS)}(\hat{\theta}_2)$	0.4871	0.6191	0.8499	0.2953	0.4020	0.6149
	$R_{(L)}(\hat{\theta}_2)$	1.8337	1.6079	0.9151	1.1619	1.0658	0.6533
	$R_{(BL)}(\hat{\theta}_2)$	0.3450	0.4406	0.6369	0.2317	0.3170	0.4839

Table 4. Risk for the Bayes estimators $\hat{\theta}_3$ and $\hat{\theta}_4$ under the SELF and the LLF.

$\theta = 04$		$n = 05$			$n = 10$		
$a \downarrow$	$\beta \rightarrow$	1.50	2.00	3.00	1.50	2.00	3.00
	$\alpha \rightarrow$	2.25	4.00	9.00	2.25	4.00	9.00
0.50	$R_{(S)}(\hat{\theta}_3)$	0.0441	0.0293	0.0078	0.0161	0.0129	0.0050
	$R_{(BS)}(\hat{\theta}_3)$	1.1520	0.0833	0.0124	0.6400	0.0513	0.0089
	$R_{(L)}(\hat{\theta}_3)$	0.1072	0.0683	0.0169	0.0366	0.0290	0.0108
	$R_{(BL)}(\hat{\theta}_3)$	0.0302	0.0217	0.0119	0.0141	0.0119	0.0081
1.00	$R_{(S)}(\hat{\theta}_3)$	0.0441	0.0293	0.0078	0.0161	0.0129	0.0050
	$R_{(BS)}(\hat{\theta}_3)$	1.1520	0.0833	0.0124	0.6400	0.0513	0.0089
	$R_{(L)}(\hat{\theta}_3)$	0.5374	0.3247	0.0740	0.1686	0.1310	0.0468
	$R_{(BL)}(\hat{\theta}_3)$	0.1422	0.0986	0.0512	0.0618	0.0512	0.0343
1.50	$R_{(S)}(\hat{\theta}_3)$	0.0441	0.0293	0.0078	0.0161	0.0129	0.0050
	$R_{(BS)}(\hat{\theta}_3)$	1.1520	0.0833	0.0124	0.6400	0.0513	0.0089
	$R_{(L)}(\hat{\theta}_3)$	1.5743	0.8866	0.1833	0.4438	0.3369	0.1146
	$R_{(BL)}(\hat{\theta}_3)$	0.3918	0.2584	0.1263	0.1545	0.1263	0.0822
0.50	$R_{(S)}(\hat{\theta}_4)$	0.0097	0.0075	0.0022	0.0056	0.0047	0.0020
	$R_{(BS)}(\hat{\theta}_4)$	1.5765	0.1115	0.0156	0.8224	0.0647	0.0108
	$R_{(L)}(\hat{\theta}_4)$	0.0218	0.0164	0.0046	0.0122	0.0102	0.0042
	$R_{(BL)}(\hat{\theta}_4)$	0.0149	0.0123	0.0082	0.0093	0.0082	0.0062
1.00	$R_{(S)}(\hat{\theta}_4)$	0.0082	0.0063	0.0019	0.0050	0.0042	0.0018
	$R_{(BS)}(\hat{\theta}_4)$	1.6486	0.1166	0.0162	0.8578	0.0674	0.0112
	$R_{(L)}(\hat{\theta}_4)$	0.0821	0.0615	0.0168	0.0472	0.0395	0.0160
	$R_{(BL)}(\hat{\theta}_4)$	0.0582	0.0484	0.0326	0.0368	0.0326	0.0246
1.50	$R_{(S)}(\hat{\theta}_4)$	0.0068	0.0054	0.0016	0.0044	0.0038	0.0017
	$R_{(BS)}(\hat{\theta}_4)$	1.7231	0.1219	0.0169	0.8952	0.0702	0.0116
	$R_{(L)}(\hat{\theta}_4)$	0.1729	0.1288	0.0348	0.1030	0.0857	0.0346
	$R_{(BL)}(\hat{\theta}_4)$	0.1285	0.1071	0.0726	0.0818	0.0726	0.0549

$\rho_1 = J(0, \infty, J_1)$;

where

$J(0, \infty, \xi) = \frac{(T+\beta)^{n+\alpha}}{\Gamma(n+\alpha)} \int_0^\infty e^{-(T+\beta)S} S^{n+\alpha-1} (\xi) dS$, $J_1 = \frac{2S}{t^3}\left(\exp\left(\frac{S}{t^2}\right) - 1\right)^{-1}$ and ξ be the function of S.

Similarly, the Bayes estimate of reliability function ψ_2 (say) and the hazard rate ρ_2 (say) under the LLF for the given prior are obtained by solving the given equality:

$$\psi_2 = -\frac{1}{a} \ln J\left(0, \infty, e^{-aJ_2}\right)$$

and

$$\rho_2 = -\frac{1}{a} \ln J\left(0, \infty, e^{-aJ_1}\right)$$

where $J_2 = a\left(1 - \exp\left(-\frac{S}{t^2}\right)\right)$.

The close forms of the Bayes estimates of the $\psi(t)$ and $\rho(t)$ under the LLF do not exist. The risk and Bayes risks do not exist in the closed form. However, the numerical values of the risk and the Bayes risk for these Bayes estimators under the SELF and LLF $R_{(S)}(\psi_i)$, $R_{(L)}(\psi_i)$, $R_{(BS)}(\psi_i)$, $R_{(BL)}(\psi_i)$, $R_{(S)}(\rho_i)$, $R_{(L)}(\rho_i)$, $R_{(BS)}(\rho_i)$ and $R_{(BL)}(\rho_i)$; $i = 1, 2$ are obtained numerically.
The expressions of the risk and the Bayes risks of these estimators involves n, a, $\theta\ \alpha$, t and β. Under the simulation study as considered in section 4, with the mission time $t = 7.50$ hours, we estimated all the risks and Bayes risks and summarized in the Tables 5-6, only for $n = 05, 10$ and $\theta = 04$.
The risk and Bayes risk of the Bayes estimates for the reliability and hazard rate are decreasing when sample size n or the prior parameters (α, β) are increasing (except Bayes risks of ψ_1 under both risks criterion) when others parametric values are fixed. The opposite trend has been seen when $'a'$ increases for risks and Bayes risk under the LLF criterion. Similar trend has been seen that the risk increases when θ increases under SELF and LLF criterion. With respect to magnitude, the LLF–criterion has smaller risk and Bayes risk with respect to SELF (except ψ_2 when $a = 1.50$).

Table 5. Values of the Bayes estimators for the Reliability and their risks under the SELF and the LLF

$\theta = 04$	$t = 7.50$	$n = 05$			$n = 10$		
$a \downarrow$	$\beta, \alpha \rightarrow$	1.50,2.25	2.00,4.00	3.00,9.00	1.50,2.25	2.00,4.00	3.00,9.00
0.50	ψ_1	0.0710	0.0686	0.0732	0.1171	0.1046	0.0983
	$R_{(S)}(\psi_1)$	15.624	15.609	15.542	15.565	15.561	15.519
	$R_{(BS)}(\psi_1)$	3.1524	4.8454	9.7075	3.1474	4.8398	9.7018
	$R_{(L)}(\psi_1)$	1.1149	1.1141	1.1105	1.1117	1.1115	1.1092
	$R_{(BL)}(\psi_1)$	0.2662	0.3987	0.7318	0.2660	0.3984	0.7317
1.00	ψ_1	0.0709	0.0683	0.0736	0.1166	0.1039	0.0982
	$R_{(S)}(\psi_1)$	15.624	15.609	15.542	15.565	15.561	15.519
	$R_{(BS)}(\psi_1)$	3.1524	4.8454	9.7075	3.1474	4.8398	9.7018
	$R_{(L)}(\psi_1)$	2.9719	2.9701	2.9618	2.9646	2.9641	2.9589
	$R_{(BL)}(\psi_1)$	0.7969	1.1687	2.0300	0.7965	1.1683	2.0298
1.50	ψ_1	0.0707	0.0682	0.0732	0.1173	0.1039	0.0988
	$R_{(S)}(\psi_1)$	15.624	15.609	15.542	15.565	15.561	15.519
	$R_{(BS)}(\psi_1)$	3.1524	4.8454	9.7075	3.1474	4.8398	9.7018
	$R_{(L)}(\psi_1)$	4.9317	4.9289	4.9163	4.9205	4.9198	4.9118
	$R_{(BL)}(\psi_1)$	1.4266	2.0593	3.4532	1.4261	2.0589	3.4531
0.50	ψ_2	1.9294	1.9308	1.9306	1.8835	1.8948	1.8882
	$R_{(S)}(\psi_2)$	4.2325	4.2018	4.1992	4.2254	4.1924	4.1712
	$R_{(BS)}(\psi_2)$	2.1303	1.2390	1.0186	2.1149	1.2344	1.0117
	$R_{(L)}(\psi_2)$	0.3861	0.3837	0.3835	0.3746	0.3830	0.3813
	$R_{(BL)}(\psi_2)$	0.1910	0.1824	0.1173	0.1904	0.1235	0.1032
1.00	ψ_2	1.9290	1.9310	1.9285	1.8834	1.8950	1.8950
	$R_{(S)}(\psi_2)$	4.2338	4.2007	4.1958	4.1939	4.1978	4.1876
	$R_{(BS)}(\psi_2)$	2.1320	1.2016	1.0107	2.1179	1.1289	1.0060
	$R_{(L)}(\psi_2)$	1.1873	1.1853	1.1783	1.1777	1.1797	1.1776
	$R_{(BL)}(\psi_2)$	0.8144	0.5927	0.4587	0.8025	0.5906	0.4475
1.50	ψ_2	1.9283	1.9308	1.9276	1.8828	1.8948	1.897
	$R_{(S)}(\psi_2)$	4.2345	4.2006	4.1949	4.1779	4.1129	4.1271
	$R_{(BS)}(\psi_2)$	2.1334	1.2064	1.0117	2.1039	1.1435	1.0041
	$R_{(L)}(\psi_2)$	2.1322	2.1205	2.1185	2.1170	2.1147	2.1027
	$R_{(BL)}(\psi_2)$	2.3434	1.1301	1.0875	2.2887	1.1195	1.0672

Table 6. Values of the Bayes estimators for the Hazard function and their risks under the SELF and the LLF

$\theta = 04$	$t = 7.50$	$n = 05$			$n = 10$		
$a \downarrow$	$\beta, \alpha \rightarrow$	1.50,2.25	2.00,4.00	3.00,9.00	1.50,2.25	2.00,4.00	3.00,9.00
0.50	ρ_1	0.2570	0.2573	0.2561	0.2505	0.2524	0.2549
	$R_{(S)}(\rho_1)$	13.998	13.987	13.984	12.007	12.001	11.977
	$R_{(BS)}(\rho_1)$	8.5409	4.0229	2.5377	8.5341	4.0159	2.1538
	$R_{(L)}(\rho_1)$	1.0247	1.0241	1.0239	1.0152	1.0149	1.0135
	$R_{(BL)}(\rho_1)$	0.6567	0.3361	0.2163	0.6561	0.3352	0.2160
1.00	ρ_1	0.2569	0.2574	0.2562	0.2504	0.2523	0.2547
	$R_{(S)}(\rho_1)$	13.998	13.987	13.984	12.001	12.007	11.977
	$R_{(BS)}(\rho_1)$	8.5409	4.0229	2.5377	8.5341	4.0159	2.2538
	$R_{(L)}(\rho_1)$	2.7651	2.7637	2.7633	2.2663	2.2655	2.2623
	$R_{(BL)}(\rho_1)$	1.8420	0.9950	0.6521	1.8402	0.9921	0.6514
1.50	ρ_1	0.2570	0.2574	0.2561	0.2506	0.2523	0.2548
	$R_{(S)}(\rho_1)$	13.998	13.987	13.984	12.007	12.001	11.977
	$R_{(BS)}(\rho_1)$	8.5409	4.0229	2.5377	8.5341	4.0159	2.1538
	$R_{(L)}(\rho_1)$	4.6156	4.6135	4.6129	4.5176	4.5163	4.5115
	$R_{(BL)}(\rho_1)$	3.1536	1.7654	1.1744	3.1505	1.7599	1.1618
0.50	ρ_1	0.2570	0.2567	0.2604	0.2505	0.2514	0.2400
	$R_{(S)}(\rho_1)$	13.999	13.992	13.990	12.170	12.937	11.893
	$R_{(BS)}(\rho_1)$	8.5204	4.0567	2.6849	7.5881	3.2094	2.0200
	$R_{(L)}(\rho_1)$	1.0278	1.0244	1.0242	1.0241	1.0212	1.0187
	$R_{(BL)}(\rho_1)$	0.6599	0.3565	0.2799	0.6554	0.3404	0.2339
1.00	ρ_1	0.2570	0.2570	0.2587	0.2505	0.2518	0.2464
	$R_{(S)}(\rho_1)$	13.993	13.993	13.990	12.082	11.965	11.943
	$R_{(BS)}(\rho_1)$	8.5278	4.0319	2.5701	2.5527	3.0794	2.4649
	$R_{(L)}(\rho_1)$	2.7844	2.7644	2.7641	2.7758	2.7608	2.7579
	$R_{(BL)}(\rho_1)$	1.8446	1.0164	0.7143	1.8401	0.9997	0.6671
1.50	ρ_2	0.2570	0.2571	0.2579	0.2504	0.2519	0.2486
	$R_{(S)}(\rho_2)$	13.994	13.990	13.989	12.054	11.975	11.960
	$R_{(BS)}(\rho_2)$	8.5309	4.0272	2.5491	7.5458	3.0527	1.5963
	$R_{(L)}(\rho_2)$	4.6349	4.6142	4.6140	4.6267	4.6110	4.6081
	$R_{(BL)}(\rho_2)$	3.1560	1.7881	1.2339	3.1516	1.7710	1.1851

6. Predictive Density and Prediction Limits

6.1 Case 01: One–Sample Bayesian Prediction Technique

Let $x_1, x_2, ..., x_n$ be a random sample of size n drawn from the model (1). Suppose m units of the same kind are to be put into future use and let $Y = (y_1, y_2, ..., y_m)$ be a second independent random sample of future observations from same model. Then the Bayesian predicative density of Y is denote by $h(y\,|\,\underline{x})$ and obtained by simplifying:

$$h(y\,|\,\underline{x}) = \int_\theta f(y;\theta)\,Z(\theta)\,d\theta = \frac{2(\alpha+n)\left(T+\beta\right)^{\alpha+n}}{y^3\left(T+\beta+y^{-2}\right)^{\alpha+n+1}}. \tag{7}$$

In the context of prediction, we say that (l_1, l_2) is a $100(1-\varepsilon)\%$ prediction interval for the future random variable Y if:

$$\Pr\left(l_1 < Y < l_2\right) = 1-\varepsilon, \tag{8}$$

where l_1 and l_2 are lower and upper prediction limits for the random variable Y and $1-\varepsilon$ is called the confidence prediction coefficient. If we consider equal tail limits, (8) become:

$$\Pr\left(Y \le l_1\right) = \frac{\varepsilon}{2} = \Pr\left(Y \ge l_2\right). \tag{9}$$

Using (7) and (9), the lower and upper prediction limits of the random variable Y are obtained as:

$$l_1 = \left\{ \left(T+\beta\right)\left(\left(\frac{2}{\varepsilon}\right)^{1/(n+\alpha)} - 1 \right) \right\}^{-1/2}$$

and

$$l_2 = \left\{ \left(T+\beta\right)\left(\left(\frac{2}{2-\varepsilon}\right)^{1/(n+\alpha)} - 1 \right) \right\}^{-1/2}. \tag{10}$$

Here n, θ, α, β and ε are involve in the expressions of l_1 and l_2. Under simulation study as considered in section 4, the limits have been calculated and presented them in Table 7 for the similar set of values as considered earlier with the confidence level $\varepsilon = 99\%, 95\%, 90\%$. It is noted that the Bayes predictive length of the interval $(l_2 - l_1)$ tends to be closer when ε increases and widens as θ increases. Further, the predictive interval also tends to be closer when sample size n increases.

Table 7. Bayes predictive length of the interval under One–Sample Bayes Prediction Technique

$\beta, \alpha \downarrow$	$\varepsilon \rightarrow$	99%		95%		90%	
$\theta = 04$	n	Lower	Upper	Lower	Upper	Lower	Upper
1.50, 2.25	5	1.2546	3.7184	1.5541	3.6094	1.7323	3.4630
	10	1.0265	3.0996	1.2741	3.0290	1.4357	2.9073
	15	0.7287	2.3777	0.9304	2.2963	1.0539	2.2116
2.00, 4.00	5	1.1750	3.4384	1.4464	3.3380	1.6071	3.2056
	10	0.7440	2.3398	0.9369	2.2642	1.0618	2.1893
	15	0.9825	2.9250	1.2152	2.8582	1.3652	2.7629
3.00, 9.00	5	1.1151	3.2232	1.3584	3.1238	1.5285	3.0208
	10	0.8181	2.4449	1.0116	2.3700	1.1360	2.2816
	15	0.9779	2.8588	1.1986	2.7741	1.3431	2.6757
$\theta = 06$							
1.50, 2.25	5	1.2858	3.8026	1.5809	3.6842	1.7793	3.5603
	10	0.7443	2.4269	0.9526	2.3584	1.0838	2.2737
	15	1.0527	3.1892	1.3087	3.0869	1.4754	2.9855
2.00, 4.00	5	1.1962	3.5004	1.4710	3.3933	1.6445	3.2794
	10	0.7594	2.3808	0.9543	2.3145	1.0807	2.2304
	15	1.0033	2.9917	1.2380	2.9045	1.3922	2.8008
3.00, 9.00	5	1.1307	3.2583	1.3790	3.1698	1.5435	3.0601
	10	0.8278	2.4760	1.0237	2.4037	1.1509	2.3194
	15	0.9906	2.8934	1.2143	2.8115	1.3598	2.7127
$\theta = 08$							
1.50, 2.25	5	1.3069	3.8455	1.6149	3.7269	1.7919	3.6029
	10	0.7563	2.4566	0.9626	2.3824	1.0973	2.2993
	15	1.0677	3.2302	1.3224	3.1370	1.4896	3.0200
2.00, 4.00	5	1.2080	3.5309	1.4795	3.4321	1.6592	3.3129
	10	0.7660	2.4063	0.9637	2.3376	1.0911	2.2497
	15	1.0090	3.0215	1.2459	2.9335	1.4043	2.8268
3.00, 9.00	5	1.1384	3.2813	1.3875	3.1888	1.5554	3.0771
	10	0.8339	2.4886	1.0308	2.4196	1.1575	2.3349
	15	0.9986	2.9124	1.2225	2.8307	1.3683	2.7322

6.2 Case 02: Two–Sample Bayesian Prediction Technique

Since $x_1, x_2, ..., x_r$ are the first r components from a sample of size n under the (1). If $y_1, y_2, ..., y_m$ is the second (unobserved) data of size m drawn independently from the sample of size N of the same model, then the first sample is referred to as the informative (past) sample, while the second one is referred to as the (future) sample. Based on an informative sample, our aim is to predict the j^{th} order statistic in the future sample.

Using the predictive density (7) of the future observation Y, the cumulative predictive density function is obtain as:

$$G\left(y \mid \underline{x}\right) = \Pr\left(Y \le y\right) = \left(1 + \frac{1}{\left(T+\beta\right) y^2}\right)^{-n-\alpha}. \tag{11}$$

Now, if Y_j be the j^{th} order statistic in the future sample of size $m, 1 \le j \le m$, then the probability density function of the j^{th} ordered future observation from the m future observations is obtain as:

$$\phi(y_j) = j\binom{m}{j}\left(G\left(y \mid \underline{x}\right)\right)^{j-1}\left(1 - G\left(y \mid \underline{x}\right)\right)^{m-j} h\left(y \mid \underline{x}\right). \tag{12}$$

To find the prediction limits for Y_j, the j^{th} smallest observation from a set of m future observations with probability density function (12), we choose l_{1j} and l_{2j} such as,

$$\Pr\left(l_{1j} < Y_j < l_{2j}\right) = 1 - \varepsilon. \tag{13}$$

Using the equation (11), (12) and (9), the expressions of the limits for the j^{th} future observations are obtained by solving:

$$j\binom{m}{j}\int_0^{\hat{l}_1} Z^{j-1}(1-Z)^{(m-j)}\, dZ = \frac{\varepsilon}{2}$$

and

$$j\binom{m}{j}\int_0^{\hat{l}_2} Z^{j-1}(1-Z)^{(m-j)}\, dZ = 1 - \frac{\varepsilon}{2}, \tag{14}$$

where $\hat{l}_i = \left(1 + \frac{1}{\left(T+\beta\right) l_{ij}^2}\right)^{-n-\alpha}, i = 1, 2, \; j = 1, 2, ..., m.$

Solving (14) for $j = 1,$ the lower and upper prediction limits of the first future observation are given as

$$l_{11} = \left(\left(T+\beta\right)\left(Z_1 - 1\right)\right)^{-1/2}$$

and

$$l_{21} = \left(\left(T+\beta\right)\left(Z_2 - 1\right)\right)^{-1/2}; \tag{15}$$

where $Z_1 = \left(1 - \left(1 - \frac{\varepsilon}{2}\right)^{1/m}\right)^{-1/(n+\alpha)}$ and $Z_2 = \left(1 - \left(\frac{\varepsilon}{2}\right)^{1/m}\right)^{-1/(n+\alpha)}$.

Similarly, solving the equation (14) for $j = m,$ to obtained the prediction limits for the last future observation as:

$$l_{1m} = \sqrt{\frac{1}{T+\beta}\left(\left(\frac{\varepsilon}{2}\right)^{-1/m(n+\alpha)} - 1\right)}$$

and

$$l_{2m} = \sqrt{\frac{1}{T+\beta}\left(\left(1-\frac{\varepsilon}{2}\right)^{-1/m(n+\alpha)} - 1\right)}. \tag{16}$$

Hence, the Bayesian prediction length of intervals for the smallest (first one) and the largest (last one) future observation are given as

$$I_{(F)} = l_{21} - l_{11} \text{ and } I_{(L)} = l_{2m} - l_{1m}. \tag{17}$$

Here $n, \theta, \alpha, \beta, m$ and ε are involve in the expressions of $I_{(F)}$ and $I_{(L)}$. The limits have been calculated for the similar set of values as considered earlier and presented them in Table 8. The behaviors of the prediction intervals are similar to One–Sample plan. Further, the intervals tend be wider when m increases.

This is natural, since the prediction of the future order statistic that is far away from the last observed value and has less accuracy than that of other future order statistic.

Table 8. Bayes predictive length of the interval under Two–Sample Bayes Prediction Technique

			First Future Observation			Last Future Observation		
n	θ	β, α	ε = 99%	ε = 95%	ε = 90%	ε = 99%	ε = 95%	ε = 90%
05	04	1.50, 2.25	0.7289	0.5399	0.4452	0.8339	0.6498	0.5488
		2.00, 4.00	1.0873	0.8012	0.6573	1.2655	0.9609	0.8123
		3.00, 9.00	1.3573	0.9953	0.8258	1.5727	1.2006	1.0101
	06	1.50, 2.25	0.7806	0.5648	0.4693	0.8937	0.6850	0.5782
		2.00, 4.00	1.1220	0.8232	0.6786	1.2926	0.9866	0.8324
		3.00, 9.00	1.3854	1.0145	0.8380	1.5956	1.2163	1.0229
	08	1.50, 2.25	0.7888	0.5804	0.4780	0.9143	0.6977	0.5868
		2.00, 4.00	1.1343	0.8296	0.6844	1.3074	0.9955	0.8386
		3.00, 9.00	1.3903	1.0177	0.8414	1.6053	1.2236	1.0302
10	04	1.50, 2.25	0.5119	0.3797	0.3171	0.6114	0.4712	0.4010
		2.00, 4.00	0.7582	0.5590	0.4655	0.9078	0.6995	0.5910
		3.00, 9.00	0.9414	0.6954	0.5788	1.1262	0.8690	0.7367
	06	1.50, 2.25	0.5362	0.3963	0.3305	0.6451	0.4976	0.4198
		2.00, 4.00	0.7766	0.5748	0.4767	0.9299	0.7165	0.6058
		3.00, 9.00	0.9557	0.7080	0.5874	1.1457	0.8837	0.7468
	08	1.50, 2.25	0.5459	0.4044	0.3361	0.6558	0.5058	0.4280
		2.00, 4.00	0.7816	0.5788	0.4809	0.9370	0.7227	0.6115
		3.00, 9.00	0.9611	0.7118	0.5901	1.1518	0.8881	0.7514

7. Conclusion

In the present paper the properties of the Bayes estimates of the parameter, reciprocal of the parameter, reliability function and hazard rate have been studied for the inverse Rayleigh model. The risk and Bayes risks do not exist in the closed form for the reliability function and hazard rate. However, the numerical values of the risk and the Bayes risk for these Bayes estimators under the SELF and LLF criterion are obtained numerically.
We also predict the future order statistic based on the observed ordered statistic and obtain the prediction intervals for unobserved order statistic under One and Two–Sample prediction technique.

References

[1]. Ahmad, A.A., Mohammad, Z. R. and Mohamed, T. M. (2007). Bayesian prediction intervals for the future order statistics from the generalized Exponential distribution. *JIRSS*, 6(1), 17–30.

[2]. Bain, L. J. (1978). *Statistical analysis of reliability and life testing model*. Marcel Dekker, New York.

[3]. Banerjee, A. K. and Bhattacharya, G. K. (1979). Bayesian results for the inverse Gaussian distribution with an application. *Technometrics*, 21, 247–251.

[4]. Cramer, E. and Kamps, U. (1996). Sequential order statistics and k–out–of–n systems with sequentially adjusted failure rates. *Annals of the Institute of Statistical Mathematics*, 48, 535–549.

[5]. Fernandez, A. J. (2000). Bayesian inference from Type–II doubly censored Rayleigh data. *Statistical Probability Letter*, 48, 393–399.

[6]. Howlader, H. A. (1985). HPD prediction intervals for Rayleigh distribution. *IEEE Transaction on Reliability*, R–34, 121–123.

[7]. Mousa, M. A. M. Ali and Al–Sagheer, S. A. (2005). Bayesian prediction for progressively Type–II censored data from Rayleigh model. *Communication in Statistics–Theory and Methods*, 34, 2353–2361.

[8]. Nigm, A. M., AL–Hussaini, E. K. and Jaheen, Z. F. (2003). Bayesian one–sample prediction of future observations under Pareto distribution. *Statistics*, 37 (6), 527–536.

[9]. Prakash, G. and Singh, D. C. (2006). Shrinkage testimators for the inverse dispersion of the inverse Gaussian distribution under the LINEX loss function. *Austrian Journal of Statistics*, 35 (4), 463–470.

[10]. Prakash, G. and Singh, D. C. (2008). Shrinkage Estimation in Exponential Type–II Censored data Under the LINEX Loss function. *Journal of the Korean Statistical Society*, 37 (1), 53–61.

[11]. Singh, D. C., Prakash, G. and Singh, P. (2007). Shrinkage Testimators for the Shape Parameter of Pareto Distribution Using the LINEX Loss Function. *Communication in Statistics Theory and Methods*, 36 (4), 741–753.

[12]. Sinha, S. K. (1985). Bayes estimation of reliability function of normal life distribution. *IEEE Transactions on Reliability*, R–34, 1, 60–64.

[13]. Sinha, S. K. (1990). On the prediction limits for Rayleigh life distribution. *Calcutta Statistical Association Bulletin*, 39,105–109.
[14]. Soland, R. M. (1969). Bayesian analysis of the Weibull process with unknown scale and shape parameters. *IEEE Transaction on Reliability*, R–18, 181–184.
[15]. Son, Y. S. and Oh, M. (2006). Bayesian estimation of the two–parameter Gamma distribution. *Communications in Statistics–Simulation and Computation*, 35, 285–293.
[16]. Srivastava, R. and Tanna, V. (2001). An estimation procedure for error variance incorporating PTS for random effects model under LINEX loss function. *Communications in Statistics–Theory and Methods*, 30, 2583–2599.
[17]. Raqab, M. Z. (1997). Modified maximum likelihood predictors of future order statistics from normal samples. *Computational Statistics and Data Analysis*, 25, 91–106.
[18]. Raqab, M. Z. and Madi, M. T. (2002). Bayesian prediction of the total time on test using doubly censored Rayleigh Data. *Journal of Statistical Computational and Simulation*, 72, 781–789.
[19]. Varian, H.R (1975). *A Bayesian approach to real estate assessment. In studies in Bayesian econometrics and statistics in honor of L. J. Savage, Eds S. E. Feinberge and A. Zellner*. Amsterdam, North Holland, 195–208.
[20]. Xu, Y. and Shi, Y. (2004). Empirical Bayes test for truncation parameters using LINEX loss. *Bulletin of the Institute of Mathematics Academia SINICA*, 32 (3), 207–220.
[21]. Zellner, A. (1986). Bayesian estimation and prediction using asymmetric loss function. *Journal of the American Statistical Association*, 81, 446–451.